Group algebra $A(O)$ and the local gauge symmetry of quantum space

Yingming Xiang*

(Dated: June 27, 2017)

PACS numbers: 12.10.-g,11.15.-q,

Keywords: group algebra,gauge field,quantum spacetime,interactions,particles,gravity

Yang-Mills local gauge field theory[1] is basic theory of the particle physics. In four dimensional relativity space-time$(\overrightarrow{r}, t)$, the wave function $\Psi(\overrightarrow{r}, t)$ describes the particles.Describing the particles and their interactions are sum up to let the wave function $\Psi(\overrightarrow{r}, t)$ meet local gauge transformation,

$$\Psi'(\overrightarrow{r}, t) = \exp(-i\theta^{\alpha}(\overrightarrow{r}, t)T_{\alpha})\Psi(\overrightarrow{r}, t). \tag{1}$$

Here, (1) the *local* means that the phase angle θ^{α} is a function of spacetime $(\overrightarrow{r}, t)$. (2) T_{α} are the generators,which is elements of the some Lie algebra. The representations of these algebras are the element particles $\Psi(\overrightarrow{r}, t)$, and the gauge field of this local gauge symmetry are their interactions. (3) The mass of particles can be obtained by using the Higgs mechanism[2, 3] of spontaneous symmetry breaking.

In general, these generators are derived from the $SU(N)$ symmetries of the internal space. For example,in the Gell-Mann quark*(uds)* theory[4–7],to describe the flavor space,the symmetry is $SU(3)$, the generators are $T_{\alpha} \in A_2$,element particle states are (uds), the corresponding eight gauge field, namely the flavor interactions. If we use the subgroup $SU(2) \otimes U^Y(1)$ of $SU(3)$ to describe the flavor,their corresponding four gauge fields of generators are $(W^{\pm}, Z^0, \gamma)$,which is just the electroweak interactions in the GWS's Standard Model[8–10].To describe the color space, the symmetry is another $SU(3)$, the generators are another $T_{\alpha} \in A_2'$, its basic representation are (RGB), the corresponding gauge fields are eight $G^{\mu\nu}$, which are the strong interactions.To describe the spin space, the symmetry is $SU(2)$, the generators are $T_{\alpha} \in A_1$,its basic representation are $(-\frac{1}{2}, +\frac{1}{2})$, the corresponding gauge field is considering in spin properties of other gauge fields[11]. Plus the Higgs mechanism of spontaneous symmetry breaking, it will be able to describe all quark*(uds)* particles, which internal space are the spin space, flavor space and color space, their symmetries are $SU^{spin}(2) \otimes SU^{flavor}(3) \otimes SU^{color}(3)$, their generators meet the Lie algebra $A_1^{spin} \oplus A_2^{flavor} \oplus A_2^{color}$. Ignore the physical significance of these Lie algebra, they have the Lie algebraic structure of $A_1 \oplus A_2 \oplus A_2'$. Here the structure of A_2 and A_2' are same, but their corresponding generators are different.

Actually, according to the local gauge transformation Formula (1),only T_{α} are need to know.If knowing generators T_{α},it's easy to get representations space and corresponding gauge fields of T_{α}.In theory, we can try any other generators compatible with the experiment,for example,the Standard Model. Here, I introduce a set of the special generators: the group algebra of group O .

* Department of Physics, Beijing Normal University,Beijing 100875, China; xiangymm@sohu.com

I. GROUP ALGEBRA $A(O)$ OF GROUP O

TABLE I. Corresponds between O and S_4

Symbol	Element of S_4	Element of O	Classes
$T_0 = E$	E	E	E
T_1	(12)(34)	$C_4(001)$	$C_4^2(3)$
T_2	(13)(24)	$C_4(100)$	
T_3	(12)(34)	$C_4(010)$	
T_4	(123)	$C_3(1-11)$	$C_3'(8)$
T_5	(142)	$C_3(-111)$	
T_6	(134)	$C_3(11-1)$	
T_7	(243)	$C_3(111)$	
T_8	(132)	$C_3(1-11)$	
T_9	(124)	$C_3(-111)$	
T_{10}	(143)	$C_3(11-1)$	
T_{11}	(234)	$C_3(111)$	
T_{12}	(12)	$C_2(110)$	$C_2''(6)$
T_{13}	(13)	$C_2(011)$	
T_{14}	(14)	$C_2(101)$	
T_{15}	(23)	$C_2(-101)$	
T_{16}	(24)	$C_2(01-1)$	
T_{17}	(34)	$C_2(1-10)$	
T_{18}	(1234)	$C_4(100)$	$C_4(6)$
T_{19}	(1243)	$C_4(010)$	
T_{20}	(1324)	$C_4(001)$	
T_{21}	(1432)	$C_4(100)$	
T_{22}	(1342)	$C_4(010)$	
T_{23}	(1423)	$C_4(001)$	

The symmetric group of the octahedron is called O, and O has 24 group elements, divided into five classes, as shown in the third column of Table I. Because O and the exchange group S_4 are isomorphic, they have one-to-one correspondence, see 2th and 3th columns in Table I. For convenience, we use $T_0, T_1, T_2, ..., T_{23}$ to mark these group elements (Table I). The multiplication table of S_4 is easy to find in the textbook[12]. So, the multiplication table of O is Table II.

The group algebra of O is an algebra in space with 24 dimensions $(T_0, T_1, T_2, ..., T_{23})$, denoted by *A(O)*.The multiplication of *A(O)* satisfy the multiplication table of O (Table II). So there is a definite meaning of the com-

TABLE II. Multiplication table of O

	E	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}	T_{17}	T_{18}	T_{19}	T_{20}	T_{21}	T_{22}	T_{23}
E	E	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}	T_{17}	T_{18}	T_{19}	T_{20}	T_{21}	T_{22}	T_{23}
T_1	T_1	E	T_3	T_2	T_7	T_6	T_5	T_4	T_{10}	T_{11}	T_8	T_9	T_{17}	T_{21}	T_{22}	T_{19}	T_{18}	T_{12}	T_{16}	T_{15}	T_{23}	T_{13}	T_{14}	T_{20}
T_2	T_2	T_3	E	T_1	T_5	T_4	T_7	T_6	T_{11}	T_{10}	T_9	T_8	T_{23}	T_{16}	T_{19}	T_{22}	T_{13}	T_{20}	T_{21}	T_{14}	T_{17}	T_{18}	T_{15}	T_{12}
T_3	T_3	T_2	T_1	E	T_6	T_7	T_4	T_5	T_9	T_8	T_{11}	T_{10}	T_{20}	T_{18}	T_{15}	T_{14}	T_{21}	T_{23}	T_{13}	T_{22}	T_{12}	T_{16}	T_{19}	T_{17}
T_4	T_4	T_6	T_7	T_5	T_8	T_{10}	T_{11}	T_9	E	T_2	T_3	T_1	T_{13}	T_{15}	T_{23}	T_{12}	T_{19}	T_{18}	T_{22}	T_{20}	T_{16}	T_{14}	T_{17}	T_{21}
T_5	T_5	T_7	T_6	T_4	T_{11}	T_9	T_8	T_{10}	T_2	E	T_1	T_3	T_{16}	T_{22}	T_{12}	T_{23}	T_{14}	T_{21}	T_{15}	T_{17}	T_{13}	T_{19}	T_{20}	T_{18}
T_6	T_6	T_4	T_5	T_7	T_9	T_{11}	T_{10}	T_8	T_3	T_1	E	T_2	T_{18}	T_{14}	T_{17}	T_{20}	T_{22}	T_{13}	T_{19}	T_{12}	T_{21}	T_{15}	T_{23}	T_{16}
T_7	T_7	T_5	T_4	T_6	T_{10}	T_8	T_9	T_{11}	T_1	T_3	T_2	E	T_{21}	T_{19}	T_{20}	T_{17}	T_{15}	T_{16}	T_{14}	T_{23}	T_{18}	T_{22}	T_{12}	T_{13}
T_8	T_8	T_{11}	T_9	T_{10}	E	T_3	T_1	T_2	T_4	T_7	T_5	T_6	T_{15}	T_{12}	T_{21}	T_{13}	T_{20}	T_{22}	T_{17}	T_{16}	T_{19}	T_{23}	T_{18}	T_{14}
T_9	T_9	T_{10}	T_8	T_{11}	T_3	E	T_2	T_1	T_6	T_5	T_7	T_4	T_{14}	T_{20}	T_{16}	T_{18}	T_{12}	T_{19}	T_{23}	T_{21}	T_{22}	T_{17}	T_{13}	T_{15}
T_{10}	T_{10}	T_9	T_{11}	T_8	T_1	T_2	E	T_3	T_7	T_4	T_6	T_5	T_{19}	T_{17}	T_{13}	T_{21}	T_{23}	T_{14}	T_{12}	T_{18}	T_{15}	T_{20}	T_{16}	T_{22}
T_{11}	T_{11}	T_8	T_{10}	T_9	T_2	T_1	T_3	E	T_5	T_6	T_4	T_7	T_{22}	T_{23}	T_{18}	T_{16}	T_{17}	T_{15}	T_{20}	T_{13}	T_{14}	T_{12}	T_{21}	T_{19}
T_{12}	T_{12}	T_{17}	T_{20}	T_{23}	T_{15}	T_{14}	T_{22}	T_{19}	T_{13}	T_{16}	T_{21}	T_{18}	E	T_8	T_5	T_4	T_9	T_1	T_{11}	T_7	T_2	T_{10}	T_6	T_3
T_{13}	T_{13}	T_{18}	T_{16}	T_{21}	T_{12}	T_{23}	T_{17}	T_{20}	T_{15}	T_{19}	T_{14}	T_{22}	T_4	E	T_{10}	T_8	T_2	T_6	T_1	T_9	T_7	T_3	T_{11}	T_5
T_{14}	T_{14}	T_{19}	T_{22}	T_{15}	T_{18}	T_{16}	T_{13}	T_{21}	T_{20}	T_{12}	T_{17}	T_{23}	T_9	T_6	E	T_3	T_5	T_{10}	T_4	T_1	T_8	T_7	T_2	T_{11}
T_{15}	T_{15}	T_{22}	T_{19}	T_{14}	T_{13}	T_{21}	T_{18}	T_{16}	T_{12}	T_{20}	T_{23}	T_{17}	T_8	T_4	T_3	E	T_7	T_{11}	T_6	T_2	T_9	T_5	T_1	T_{10}
T_{16}	T_{16}	T_{21}	T_{13}	T_{18}	T_{23}	T_{12}	T_{20}	T_{17}	T_{22}	T_{14}	T_{19}	T_{15}	T_5	T_2	T_9	T_{11}	E	T_7	T_3	T_{10}	T_6	T_1	T_8	T_4
T_{17}	T_{17}	T_{12}	T_{23}	T_{20}	T_{19}	T_{22}	T_{14}	T_{15}	T_{21}	T_{18}	T_{13}	T_{16}	T_1	T_{10}	T_6	T_7	T_{11}	E	T_9	T_4	T_3	T_8	T_5	T_2
T_{18}	T_{18}	T_{13}	T_{21}	T_{16}	T_{20}	T_{17}	T_{23}	T_{12}	T_{14}	T_{22}	T_{15}	T_{19}	T_6	T_3	T_{11}	T_9	T_1	T_4	T_2	T_8	T_5	E	T_{10}	T_7
T_{19}	T_{19}	T_{14}	T_{15}	T_{22}	T_{21}	T_{13}	T_{16}	T_{18}	T_{17}	T_{23}	T_{20}	T_{12}	T_{10}	T_7	T_2	T_1	T_4	T_9	T_5	T_3	T_{11}	T_6	E	T_8
T_{20}	T_{20}	T_{23}	T_{12}	T_{17}	T_{14}	T_{15}	T_{19}	T_{22}	T_{18}	T_{21}	T_{16}	T_{13}	T_3	T_9	T_7	T_6	T_8	T_2	T_{10}	T_5	T_1	T_{11}	T_4	E
T_{21}	T_{21}	T_{16}	T_{18}	T_{13}	T_{17}	T_{20}	T_{12}	T_{23}	T_{19}	T_{15}	T_{22}	T_{14}	T_7	T_1	T_8	T_{10}	T_3	T_5	E	T_{11}	T_4	T_2	T_9	T_6
T_{22}	T_{22}	T_{15}	T_{14}	T_{19}	T_{16}	T_{18}	T_{21}	T_{13}	T_{23}	T_{17}	T_{12}	T_{20}	T_{11}	T_5	T_1	T_2	T_6	T_8	T_7	E	T_{10}	T_4	T_3	T_9
T_{23}	T_{23}	T_{20}	T_{17}	T_{12}	T_{22}	T_{19}	T_{15}	T_{14}	T_{16}	T_{13}	T_{18}	T_{21}	T_2	T_{11}	T_4	T_5	T_{10}	T_3	T_8	T_6	E	T_9	T_7	T_1

mutator $[T_\alpha, T_\beta] = T_\alpha T_\beta - T_\beta T_\alpha$. For example, we know from Table II,$T_{10}T_{15} = T_{21}, T_{15}T_{10} = T_{23}, [T_{10}, T_{15}] = T_{21} - T_{23}$. In general,$[T_\alpha, T_\beta] = \sum_{\gamma=0}^{23} C_{\alpha\beta}^{\gamma} T_\gamma$,where $C_{\alpha\beta}^{\gamma}$ are the structure constant,which given by the multiplication table of O in Table II.

We found that $A(O)$ is Lie algebra, and $A(O)$ could be decomposed into direct sum of five Lie algebra A_0, one Lie algebra A_1, one Lie algebra A_2 and another Lie algebra A_2'[11] ,

$$A(O) = \sum_{i=1}^{5} \oplus A_0^i \oplus A_1 \oplus A_2 \oplus A_2'. \tag{2}$$

The relation between the standard basis of these subalgebras and elements of $A(O)$ is as follows.

(1) The standard basis of A_0^i, denoted by X_i, $(i = 1,2,3,4,5)$.They are the sum of element in the same class.

$$\begin{cases} X_1 = T_0 = E, \\ X_2 = T_1 + T_2 + T_3, \\ X_3 = T_4 + T_5 + T_6 + T_7 + T_8 + T_9 + T_{10} + T_{11}, \\ X_4 = T_{12} + T_{13} + T_{14} + T_{15} + T_{16} + T_{17}, \\ X_5 = T_{18} + T_{19} + T_{20} + T_{21} + T_{22} + T_{23}. \end{cases} \tag{3}$$

(2) The standard basis of A_1 are $(A, E_\pm)$,

$$\begin{cases} A = i\frac{\sqrt{3}}{24}(T_4 + T_5 + T_6 + T_7 - T_8 - T_9 - T_{10} - T_{11}), \\ E_\pm = \frac{\sqrt{3}}{4}[2(T_{12} + T_{17} + T_{20} + T_{23}) \\ -(T_{13} + T_{16} + T_{18} + T_{21}) - (T_{14} + T_{175} + T_{19} + T_{22})] \\ \pm i\frac{3}{4}[(T_{13} + T_{16} + T_{18} + T_{21}) - (T_{14} + T_{15} + T_{19} + T_{22})]. \end{cases} \tag{4}$$

(3) The standard basis of A_2 are $(H_1, H_2, E_{\pm\alpha}, E_{\pm\beta}, E_{\pm(\alpha+\beta)})$,

$$\begin{cases} H_1 = \frac{1}{8\sqrt{3}}\{T_1 - T_2 \\ +\frac{1}{2}[(T_{12} + T_{17} - T_{20} - T_{23}) - (T_{14} + T_{15} - T_{19} - T_{22})]\}, \\ H_2 = \frac{1}{32}\{2T_2 - T_1 - T_3 + \frac{1}{2}[2(T_{13} + T_{16} - T_{18} - T_{21}) \\ -(T_{12} + T_{17} - T_{20} - T_{23}) - (T_{14} + T_{15} - T_{19} - T_{22})]\}, \\ E_\alpha = \frac{1}{8\sqrt{6}}[(T_4 + T_5 - T_6 - T_7) + (T_{12} - T_{17} - T_{20} + T_{23})], \\ E_{-\alpha} = \frac{1}{8\sqrt{6}}[(T_8 + T_9 - T_{10} - T_{11}) + (T_{12} - T_{17} + T_{20} - T_{23})], \\ E_\beta = -\frac{1}{8\sqrt{6}}[(T_4 - T_5 - T_6 + T_7) - (T_{14} - T_{15} - T_{19} + T_{22})], \\ E_{-\beta} = -\frac{1}{8\sqrt{6}}[(T_8 - T_9 - T_{10} + T_{11}) - (T_{14} - T_{15} + T_{19} - T_{22})], \\ E_{+(\alpha+\beta)} = \frac{1}{8\sqrt{6}}[(T_4 - T_5 + T_6 - T_7) + (T_{13} - T_{16} - T_{18} + T_{21})], \\ E_{-(\alpha+\beta)} = \frac{1}{8\sqrt{6}}[(T_8 - T_9 + T_{10} - T_{11}) + (T_{13} - T_{16} + T_{18} - T_{21})]. \end{cases} \tag{5}$$

(4) The standard basis of A_2' are

$(H_1^{'},H_2^{'},E_{\pm\alpha}^{'},E_{\pm\beta}^{'},E_{\pm(\alpha+\beta)}^{'})$,

$$\begin{cases} H_1^{'} = \frac{1}{8\sqrt{3}}\{T_1 - T_2 \\ -\frac{1}{2}[(T_{12}+T_{17}-T_{20}-T_{23})-(T_{14}+T_{15}-T_{19}-T_{22})]\}, \\ H_2^{'} = \frac{1}{32}\{2T_2 - T_1 - T_3 - \frac{1}{2}[2(T_{13}+T_{16}-T_{18}-T_{21}) \\ -(T_{12}+T_{17}-T_{20}-T_{23})-(T_{14}+T_{15}-T_{19}-T_{22})]\}, \\ E_{\alpha}^{'} = \frac{1}{8\sqrt{6}}[(T_4+T_5-T_6-T_7)-(T_{12}-T_{17}-T_{20}+T_{23})], \\ E_{-\alpha}^{'} = \frac{1}{8\sqrt{6}}[(T_8+T_9-T_{10}-T_{11})-(T_{12}-T_{17}+T_{20}-T_{23})], \\ E_{\beta}^{'} = -\frac{1}{8\sqrt{6}}[(T_4-T_5-T_6+T_7)+(T_{14}-T_{15}-T_{19}+T_{22})], \\ E_{-\beta}^{'} = -\frac{1}{8\sqrt{6}}[(T_8-T_9-T_{10}+T_{11})+(T_{14}-T_{15}+T_{19}-T_{22})], \\ E_{+(\alpha+\beta)}^{'} = \frac{1}{8\sqrt{6}}[(T_4-T_5+T_6-T_7)-(T_{13}-T_{16}-T_{18}+T_{21})], \\ E_{-(\alpha+\beta)}^{'} = \frac{1}{8\sqrt{6}}[(T_8-T_9+T_{10}-T_{11})-(T_{13}-T_{16}+T_{18}-T_{21})]. \end{cases} \tag{6}$$

Suppose that there is a Yang-Mills local gauge field (Formula 1),whose generators satisfy $A(O)$. Where,we can correspond spin with A_1, flavor with A_2, color with A_2' and some global quantum number with A_0^i , namely

$$T_\alpha \in A(O) = \sum_{i=1}^{5} \oplus A_0^i \oplus A_1^{spin} \oplus A_2^{flavor} \oplus A_2^{color}. \tag{7}$$

It provides us another possibility with a unified description of Gell-Mann quarks(uds) . The representation of these sub-algebras constitute internal space.*A(O)* is equivalent to the Lie algebra of symmetry group $\prod_{i=1}^{5} \otimes U(1)^i \otimes SU(2) \otimes SU(3) \otimes SU(3)'$, but it is not originated from this direct product group.So this is not in the traditional sense of the internal space symmetry.

Because O is a point group, what is the physical significance of the 24-dim group algebra? What is symmetry of this algebra? I guess, because the group O is a space crystal point group, is there a special lattice structure in space, and this algebra is the result of the symmetry of the spatial structure? To grid space means to quantize space.The symmetry should be the result of this quantum space. The following is my solution of quantum space-time .

II. SPACE LATTICE

Space consists of only one kind of cell, the cell is spherical,the diameter $l_0 \sim 10^{-18}m$[13], can be out of shape, accumulation into space with face-centered cubic close,denoted by $\mathcal{A}_1^{Lattice}$ (such an accumulation is the most closely and isotropic)[11].

A. Vaccum

The perfect $\mathcal{A}_1^{Lattice}$ is the vacuum. $\mathcal{A}_1^{Lattice}$ has two types of gaps,tetrahedral T and octahedral O, see

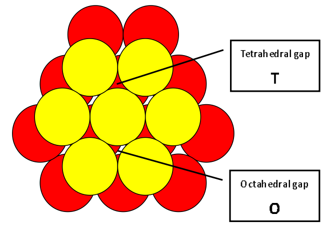

FIG. 1. Gaps of $\mathcal{A}_1^{Lattice}$

Fig.1. $\mathcal{A}_1^{Lattice}$ has three kind of symmetries.(1)For octahedral gap O, its overall point group symmetry is group $O_h = \{O, O\sigma\}$, σ is a space inversion operation.(2)For tetrahedral gap T, its overall point group symmetry is T_d.(3)Translation symmetry of $\mathcal{A}_1^{Lattice}$,$T_{\vec{l}}\vec{r} = \vec{r} + l_1\vec{e}_1 + l_2\vec{e}_2 + l_3\vec{e}_3$,where $l_i = 0,1,2,3,...; i = 1,2,3;\vec{e}_1,\vec{e}_2,\vec{e}_3$ are three unit vectors in rectangular coordinates. The whole $\mathcal{A}_1^{Lattice}$ satisfies the symmetry of the spatial group $g(R,T_{\vec{l}})$[14, 15]. Where, O_h, or T_d, are the point group; $T_{\vec{l}}$ is a translation group.

B. Matter

Defective $\mathcal{A}_1^{Lattice}$ is the matter. The symmetry of the space group $g(R,T_{\vec{l}})$ broken. But there are the following local symmetries.For any cell n, the position of the cell center is $\vec{r}_n$. When we don't think about the deformation of the cell,the set $\{\vec{r}_n\}$ of all cells can describe the state of the matter, the wave function,$\Psi(\vec{r}) = \{\overrightarrow{r}_n\}$.

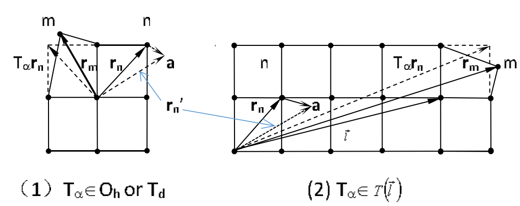

FIG. 2. Symmetry of defective $A_1^{Lattice}$

In the matter, because the spatial group symmetry broken,$g(R,T_{\vec{l}})\vec{r}_n \neq \vec{r}_m$,as shown in Figure 2. We can always find a small translation,$T_{\vec{a}}$,$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3$,$(0 \leq a_i \ll 1, i = 1,2,3)$.First,let $\vec{r}_n$ translate to $\vec{r'}_n$,namely,$\vec{r'}_n = T_{\vec{a}}\vec{r}_n$.And then ,$\vec{r'}_n$ operated by T_α,we transform the cell n to the cell m,namely $\vec{r}_m = T_\alpha\vec{r'}_n = T_\alpha T_{\vec{a}}\vec{r}_n$. (1) $T_\alpha \in g(R,T_{\vec{l}})$is a space group. When $T_\alpha \in g(R,T_{\vec{0}})$,$T_\alpha$ is in the point group O_h or T_d; When $T_\alpha \in g(E,T_{\vec{l}})$,$T_\alpha$ is in the translation group $T_{\vec{l}}$; (2) For different T_α and $\vec{r}_n$, there are different $\vec{a}$,denoted by $\vec{a}^\alpha(\vec{r}_n)$,which means localized. Without considering the cell deformation, because of the indistinguishability of the cell, such operations are able to transform all cell positions of defective $\mathcal{A}_1^{Lattice}$ to itself. Therefore,$\{\vec{r}_m\} = \{T_\alpha T_{\vec{a}} \vec{r}_n\}$ is also the same matter state. The wave function operated by the symmetric operation is

$$\Psi'(\vec{r}) = \{\vec{r}_m\} = \{T_\alpha[\vec{r}_n + \sum_{i=1}^{3} a_i^\alpha(\vec{r}_n)\vec{e}_i]\}, \tag{8}$$

its generators are[14],

$$\begin{aligned} I_i^\alpha &= -i\frac{\partial\{T_\alpha[\vec{r}_n + \sum_{i=1}^{3} a_i^\alpha(\vec{r}_n)\vec{e}_i]\}}{\partial a_i^\alpha} \mid_{a_i^\alpha=0} \frac{\partial}{\partial \vec{e}_i} \\ &= -iT_\alpha \vec{e}_i \frac{\partial}{\partial \vec{e}_i}. \end{aligned} \tag{9}$$

So

$$\begin{aligned} \Psi'(\vec{r}) &= \exp\{\sum_\alpha \sum_{i=1}^{3} a_i^\alpha(\vec{r}) I_i^\alpha\}\Psi(\vec{r}) \\ &= \exp\{-i\sum_\alpha T_\alpha[\sum_{i=1}^{3} a_i^\alpha(\vec{r})\vec{e}_i \frac{\partial}{\partial \vec{e}_i}]\}\Psi(\vec{r}). \end{aligned} \tag{10}$$

Let

$$\widehat{\theta}^\alpha(\vec{r}) = \sum_{i=1}^{3} a_i^\alpha(\vec{r})\vec{e}_i \frac{\partial}{\partial \vec{e}_i}, \tag{11}$$

we get

$$\begin{aligned} \Psi'(\vec{r}) &= \exp(-iT_\alpha \widehat{\theta}^\alpha(\vec{r}))\Psi(\vec{r}) \\ &= \exp(-iT_\alpha \theta^\alpha(\vec{r}))\Psi(\vec{r}). \end{aligned} \tag{12}$$

Here,$\theta^\alpha(\vec{r})$ is eigenvalue of operator $\widehat{\theta}^\alpha(\vec{r})$.Repeated α means the sum of all the group elements.

Above discussion is true in the space state of a certain time.It is also true for every time t,

$$\Psi'(\vec{r}, t) = \exp(-iT_\alpha \theta^\alpha(\vec{r}, t))\Psi(\vec{r}, t). \tag{13}$$

In contrast Formula (1) and (13), the space symmetry group element T_α are the generators of this local gauge symmetry, the representations of T_α are the particles, the corresponding gauge fields are the interactions.

When considering the deformation of each cell, we add a function $F(\vec{r}, t)$ to describe the shape of each cell.The first class approximation is the diameter $l(\vec{r}, t)$ of each cell,it is just Higgs scalar field which vacuum values are not zero.

There are two types of gap, O and T, in $\mathcal{A}_1^{Lattice}$.So there are two point groups O_h, T_d, and one kind of translation group in $\mathcal{A}_1^{Lattice}$.This means that there are three types of generators T_α, corresponding to three types of particles and interactions. Let's discuss them separately.

1,When $T_\alpha \in O_h$.Because $O_h = (O, \sigma O)$,and $\sigma(= \sigma T_0) \in O_h$,it means that σ is a group element.So there is space inversion symmetry in O_h. Definition $O^\pm = \frac{1}{2}(1 \pm \sigma)O$,and $P^\pm = \frac{1}{2}(1 \pm \sigma)$. Because of the relationship $(P^\pm)^2 = P^\pm$,$P^\pm$ are chiral operators. We can prove

$$A(O_h) = A(O^+) \oplus A(O^-), \tag{14}$$

$$A(O^\pm) \cong A(O). \tag{15}$$

So,the representations of $A(O_h)$ are the Gell-Mann quark(uds) with the bilateral symmetry. The gauge fields are exactly the same as the gluons $G^{\mu\nu}$ for strong interactions and the eight gauge fields for flavor interactions,which consist the four gauge $(W^\pm, Z^0, \gamma)$ of electroweak interactions[11].

2,When $T_\alpha \in T_d$. Because $T_d = (T_0, T_1, T_2, ..., T_{11}; \sigma T_{12}, \sigma T_{13}, ..., \sigma T_{23})$,$\sigma$ is not in T_d,it means that σ is not group element.So there's no space inversion symmetry in T_d. Because the group T_d and O are isomorphic, we can prove that their group algebra are isomorphic,namely[11],$A(T_d) \cong A(O)$,

$$A(T_d) = \sum_{i=1}^{5} \oplus A_0^{id} \oplus A_1^d \oplus A_2^d \oplus A_2'^d. \tag{16}$$

Let $T_{12}, T_{13}, ..., T_{23}$ in $A(O)$ replaced by $\sigma T_{12}, \sigma T_{13}, ..., \sigma T_{23}$ in Formula (3)-(6),we can get the relation between the standard basis of these sub-algebras and elements of $A(T_d)$.For example,the standard generators $(A^d, E_\pm^d)$ of A_1^d are

$$\begin{cases} A^d = i\frac{\sqrt{3}}{24}(T_4 + T_5 + T_6 + T_7 - T_8 - T_9 - T_{10} - T_{11}), \\ E_\pm^d = \frac{\sqrt{3}}{4}\sigma[2(T_{12} + T_{17} + T_{20} + T_{23}) \\ -(T_{13} + T_{16} + T_{18} + T_{21}) - (T_{14} + T_{15} + T_{19} + T_{22})] \\ \pm i\frac{3}{4}\sigma[(T_{13} + T_{16} + T_{18} + T_{21}) - (T_{14} + T_{15} + T_{19} + T_{22})]. \end{cases} \tag{17}$$

Comparing Formula (4) with (17),we have $A^d = A$, $E_\pm^d = \sigma E_\pm$.Therefore,we can get

$$\frac{1}{2}(1 - \sigma)|jm >^d = 0. \tag{18}$$

Here,$|jm >^d$ is the eigensatate of $(A^d, E_\pm^d)$.It means that the particles of $A(T_d)$ are the spin left-handed.

We have just such type particles:the leptons.The basic representations of the Lie algebra A_2^d,denoted by $(l_{\nu L}, l_L, l_L^c)$,which correspond left-handed neutrino,left-handed lepton,left-handed anti-lepton.If we choice the charge $Q = I_3 - \frac{3}{2}Y_3$,the charge of $(l_{\nu L}, l_L, l_L^c)$ are $(0, -1, +1)$.The basic representations of the Lie algebra $A_2'^d$,denoted by (e, μ, τ),correspond three kind of leptons. Plus their conjugate representations, the basic representations are just the 18 leptons[11].It's interesting that the Lagrangian form.In Yang-Mills local gauge field theory,the Lagrangian of interaction between leptons and gauge fields is

$$\pounds_{fg} = \frac{g_F}{\sqrt{2}} \begin{pmatrix} \bar{l}_{\nu L} & \bar{l}_L & \bar{l}_L^c \end{pmatrix} \gamma^\mu A_\mu \begin{pmatrix} l_{\nu L} \\ l_L \\ l_L^c \end{pmatrix}, \tag{19}$$

here

$$A_\mu = A_\mu^\alpha T_\alpha = \begin{pmatrix} \frac{A_\mu^3}{\sqrt{2}} + \frac{A_\mu^8}{\sqrt{6}} & \overline{W}_\mu^1 & \overline{W}_\mu^2 \\ W_\mu^1 & -\frac{A_\mu^3}{\sqrt{2}} + \frac{A_\mu^8}{\sqrt{6}} & \overline{W}_\mu^3 \\ W_\mu^2 & W_\mu^3 & -2\frac{A_\mu^8}{\sqrt{6}} \end{pmatrix}, \quad (20)$$

where $W_\mu^1 = \frac{A_\mu^1 + iA_\mu^2}{\sqrt{2}}$,$W_\mu^2 = \frac{A_\mu^4 + iA_\mu^5}{\sqrt{2}}$,$W_\mu^3 = \frac{A_\mu^6 + iA_\mu^7}{\sqrt{2}}$.So,there are eight gauge fields $W^{1\pm}$,$W^{2\pm}$,$W^{3\pm}$,A^3 and A^8.Because $\bar{l}_{\nu L}\gamma^\mu \overline{W}_\mu^2 l_L^c = 0$, $\bar{l}_L \gamma^\mu \overline{W}_\mu^3 l_L^c = 0$, $\bar{l}_L^c \gamma_\mu W_\mu^2 l_{\nu L} = 0$, $\bar{l}_L^c \gamma^\mu W_\mu^3 l_L = 0$,$\overline{l_L^c}\gamma^\mu A_\mu^8 l_L^c = -\overline{l_R}\gamma^\mu A_\mu^8 l_R$,Formula (20) becomes

$$A_\mu = \begin{pmatrix} \frac{A_\mu^3}{\sqrt{2}} + \frac{A_\mu^8}{\sqrt{6}} & \overline{W}_\mu^1 & 0 \\ W_\mu^1 & -\frac{A_\mu^3}{\sqrt{2}} + \frac{A_\mu^8}{\sqrt{6}} & 0 \\ 0 & 0 & -2\frac{A_\mu^8}{\sqrt{6}} \end{pmatrix}, \quad (21)$$

there are only the four gauge fields $(W^{1\pm}, A^3, A^8)$.It is like the $SU^L(2) \otimes U^R(1)$,but it is not.Writting this gauge fields in mass eigenstate of leptons,the Lagrangian (21) becomes

$$\pounds_{fg} = \frac{g_F}{\sqrt{2}} \left(\bar{l}_\nu \;\; \bar{l} \right) \gamma^\mu A'_\mu \begin{pmatrix} l_\nu \\ l \end{pmatrix}, \quad (22)$$

here

$$A'_\mu = \begin{pmatrix} \frac{1+\gamma_5}{2}\left(\frac{A_\mu^3}{\sqrt{2}} + \frac{A_\mu^8}{\sqrt{6}}\right) & \frac{1+\gamma_5}{2}\overline{W}_\mu^1 \\ \frac{1+\gamma_5}{2} W_\mu^1 & \frac{1+\gamma_5}{2}\left(-\frac{A_\mu^3}{\sqrt{2}}\right) + \frac{3-\gamma_5}{2}\frac{A_\mu^8}{\sqrt{6}}) \end{pmatrix}, \quad (23)$$

Let

$$\begin{cases} A_\mu^3 = \frac{\sqrt{g^2 + g'^2}}{g} \cos^2\theta_W Z_\mu + \frac{g'}{\sqrt{g^2 + g'^2}} A_\mu, \\ A_\mu^8 = \sqrt{3}\frac{\sqrt{g^2 + g'^2}}{g} \sin^2\theta_W Z_\mu - \sqrt{3}\frac{g'}{\sqrt{g^2 + g'^2}} A_\mu, \end{cases} \quad (24)$$

we have $g_F = g$,and the gauge field A'_μ (Formula (23)) becomes(2×2 matrix)

$$A'_\mu = \begin{pmatrix} \frac{1}{2\sqrt{2}}\frac{\sqrt{g^2+g'^2}}{g}(1+\gamma_5)Z_\mu & \frac{1+\gamma_5}{2}\overline{W}_\mu^+ \\ \frac{1+\gamma_5}{2}W_\mu^- & \{\frac{1}{2\sqrt{2}}\frac{\sqrt{g^2+g'^2}}{g}(4\sin^2\theta_W - 1 - \gamma_5)Z_\mu \\ & -\sqrt{2}\frac{g'}{\sqrt{g^2+g'^2}}A_\mu\} \end{pmatrix}, \quad (25)$$

which is just the standard model of leptons with electroweak interaction $(W^\pm, Z^0, \gamma)$,where g and g' are couple constants,θ_W is the Weinberg angle.Considering the deformation of cell diameter $l(\overrightarrow{r}, t)$ as Higgs field,the vacuum value $l_0 = 2m_W \sin\theta_W / e$.Accoding to the experiment value,$m_W = (80.385 \pm 0.015)GeV$,$\sin^2\theta_W = 0.23120 \pm 0.00015$,$e = \sqrt{4\pi/137.036}$,we get that

$$l_0 = (255.3 \pm 0.1)GeV = (7.724 \pm 0.003) \times 10^{-19} m, \quad (26)$$

which is the vacuum cell diameter. From the Higgs mass,$m_H = (1.26 \pm 0.006) \times 10^2 GeV$(ATLAS,2012),we can get,$l_H = 1.56 \times 10^{-18} m$.It means that the deformation of cell in the weak interaction is about two time of vacuum value l_0. There is a trouble of interactions between three colors (e, μ, τ) in theory, but it provide us an chance to fill in the multiple states of quarks(cbt)[11].

3,When $T_\alpha \in T_{\vec{l}}$. Because $[T_{\vec{m}}, T_{\vec{n}}] = 0$, for any translation $\vec{m}$ and $\vec{n}$,translation group is Abelian, and their group algebra is Abelian algebra. According to Yang-Mills local gauge field theory, the representation basis of Abelian generators $T_{\vec{l}}$ are one-dimensional[14],exp$(-i\vec{k} \cdot \vec{l})$,and here $\vec{k}$ is the reciprocal lattice vector,which are decided by the point group, $\vec{l}$ is the translation vector, so there are an infinite numbers of such one dimension representations, which are the free state with certain kinetic energy. The gauge fields of Abelian's generators are $A_\mu^\alpha(\vec{r}, t)$,$A_\mu(\vec{r}, t) = -ig_G A_\mu^\alpha(\vec{r}, t) T_\alpha$ as the connection $\Gamma_{\mu\nu}^\lambda$ in geometry,which is the gravitational field[11], where g_G is the coupling constant.

Conclusions.(1)The quarks*(uds)* and their interactions are originated the point group O_h of lattice $\mathcal{A}_1^{Lattice}$.(2)The leptons and their interactions are originated the point group T_d of lattice $\mathcal{A}_1^{Lattice}$.(3)The the free state with certain kinetic energy (exp$(-i\vec{k} \cdot \vec{l})$) are originated the translation symmetry of lattice $\mathcal{A}_1^{Lattice}$,their interactions is gravity.

[1] Yang C.N.and Mills R.L. *Phys. Rev.*, 96:191, 1954.
[2] Higgs P.W. *Phys.Lett.*, 12:132, 1964.
[3] Higgs P.W. *Phys.Rev.*, 145:1156, 1966.
[4] Gell-Mann M. *Phys.Rev.*, 125:1067, 1962.
[5] Gell-Mann M. *Caltech Report CTSL*, 20, 1961.
[6] Gell-Mann M. *Phys. Lett.*, 8:214, 1964.
[7] Neeman Y. *Nucl. Phys.*, 26:222, 1961.
[8] Glashow S.L. *Nucl.Phys.*, 22:579, 1961.
[9] Weinberg S. *Phys.Rev.Lett.*, 19:1264, 1967.
[10] Wald J. Salam A. *Phys.Lett.*, 13:168, 1964.
[11] Xiang Y.M. *Phys. Rev.D*, DP11613, 2016.
[12] Ka X.L. *Advanced Quantum Mechanics.* Advanced Education Press, Beijing, 1999.
[13] Lee T.D. Discrete mechanics. Lecture Notes, Erice, August, 1983.
[14] Ma Z.Q. *Group and Their Application in Physics(Chinese).* BIT Press, Beijing, 1984.
[15] Elliott J.P. and Dawber P.G. *Symmetry in Physics (Chinese).* Science Press, Beijing, 1980.
[]