Three conclusions about the primes

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Abstract

In this paper, we study the derivative of Riemann's step function, $J'(x)$. (1) Because this derivative is a real function, the real part of the nontrivial zeros of Riemann's zeta function is required to be $\frac{1}{2}$.(2) We find the step function $J^{-}(x)$ in the interval of $x \in (0,1)$, which is not 0 assumed by Riemann. (3)We introduce the density function $\rho(E,x) = \pi'(x)\pi'(E-x)$ of a real number E,its integral in E is the number of prime pairs,$N(E)$,which E can expressed as the sum of these prime pairs. We prove that $N(E)$ is always greater than 1.

In 1859, Riemann[4] got the Riemann's step function

$$J(x) = Li(x) - \sum_{\rho}[Li(x^{\rho}) + Li(x^{1-\rho})] - log2 + \int_{x}^{\infty} \frac{dt}{t(t^2-1)logt}. \quad (1)$$

where ρ is the nontrivial zeros of Riemann's ζ-function,and the sum is for all nontrivial zeros. Therefore the number of primes less than a given real number x is

$$\pi(x) = \sum_{n=1}^{M} \frac{\mu(n)}{n} J(x^{\frac{1}{n}}) = J(x) - \frac{1}{2}J(x^{\frac{1}{2}}) - \frac{1}{3}J(x^{\frac{1}{3}}) + ..., \quad (2)$$

where $x^{\frac{1}{M}} \geq 2$,$\mu(n)$ is $M\ddot{o}bius$ function.

In order to express every prime exactly, we use p_n to express every prime.The n means the n^{th} prime,and p_n is the value of the n^{th} prime.All primes is a set $P = \{p_n; n = 1,2,3,...\} = \{2,3,5,7,...\}$.According to Formula (2), $\pi(x)$ is the number of primes before x,$\pi(x) = \{0,1,2,3,4,5,...\}$.Except $\pi(x) = 0$,the $\pi(x)$ is the ordinal number of primes only less than x, that is,

$$n = \pi(x). \quad (3)$$

For example,$1 = \pi(x); x \in (2,3)$ and $2 = \pi(x); x \in (3,5)$.Then, the x satisfying the above equation is an interval,which is the inverse function π^{-1} of $\pi(x)$,

$$x = \pi^{-1}(n). \quad (4)$$

And p_n is the smallest one of these x,

$$p_n = \min\{\pi^{-1}(n)\}, \tag{5}$$

$$p_{n+1} = \min\{\pi^{-1}(n+1)\} = \max\{\pi^{-1}(n)\}. \tag{6}$$

This is the expression of primes.According to the above formula, we can calculate each prime p_n, but the process is complex.We know that $\pi(x)$ is a step function. The derivative of step function is very characteristic, which provides a way for us to study and apply $\pi(x)$. Therefore, we first study the derivative of Riemann's step function $J(x)$.

1 Derivative of J (x)

Let

$$\rho = \rho_n = \alpha_n + t_n i; (n = 1, 2, 3, ...), \tag{7}$$

are the general expression of nontrivial zeros of Riemann's ζ-function.Therefore,α_n and t_n are real number,and $0 < \alpha_n < 1$[4].Formula (1) becomes

$$J(x) = Li(x) - \sum_{n=1}^{\infty}[Li(x^{\alpha_n + t_n i}) + Li(x^{1-\alpha_n - t_n i})] - log2 + \int_x^{\infty} \frac{dt}{t(t^2-1)logt}, \tag{8}$$

here

$$Li(x) = \int_0^x \frac{dt}{logt}, \tag{9}$$

$$Li(x^\rho) = \int_0^x \frac{dt^\rho}{logt^\rho} = \int_0^x \frac{t^{\rho-1}}{logt}dt. \tag{10}$$

Therefore,$x = 1$ is a singular point and the above integral means below[2]

$$\int_0^x \frac{dt}{logt} = \lim_{\epsilon \to 0}[\int_0^{1-\epsilon} \frac{dt}{logt} + \int_{1+\varepsilon}^x \frac{dt}{logt}]. \tag{11}$$

So

$$Li'(x) = \frac{1}{logx}, \tag{12}$$

$$Li'(x^\rho) = \frac{x^{\rho-1}}{logx}. \tag{13}$$

$$Li'(x^{1-\rho}) = \frac{x^{-\rho}}{logx}. \tag{14}$$

Calculating the derivative of Formula (8) directly,

$$J'(x) = \frac{1}{logx}\{1 - \frac{1}{x(x^2-1)} - \sum_{n=1}^{\infty}[x^{\alpha_n - 1}e^{it_n logx} + x^{-\alpha_n}e^{-it_n logx}]\}. \tag{15}$$

Separate the real part and the imaginary part of the above Formula (15),

$$J'(x) = \frac{1}{logx}\{1-\frac{1}{x(x^2-1)}-\sum_{n=1}^{\infty}[(x^{\alpha_n-1}+x^{-\alpha_n})cos(t_n logx)+i(x^{\alpha_n-1}-x^{-\alpha_n})sin(t_n logx)]\}. \tag{16}$$

Because Riemann's step function $J(x)$ is a real function, its derivative is also a real function. This requires that the imaginary part of the above Formula (16) be zero,therefore

$$x^{\alpha_n-1} - x^{-\alpha_n} = 0; n = 1, 2, 3, ..., \tag{17}$$

so

$$\alpha_n = \frac{1}{2}; n = 1, 2, 3, ... \tag{18}$$

This means that the real part of all nontrivial zeros must be $\frac{1}{2}$, that is, all nontrivial zeros are on the critical line. This is what Riemann's hypothesis[4].

Therefore

$$J'(x) = \frac{1}{logx}[1 - \frac{1}{x(x^2-1)} - \frac{1}{\sqrt{x}}\sum_{n=1}^{\infty} 2cos(t_n logx)]. \tag{19}$$

Here $\rho_n^+ = \rho = \frac{1}{2}+t_n i; \rho_n^- = 1-\rho = \frac{1}{2}-t_n i$,and the symmetric form of nontrivial zeros is

$$\rho_n^{\pm} = \frac{1}{2} \pm t_n i; n = 1, 2, 3, ... \tag{20}$$

This result has been obtained by Riemann in his paper, except for items $-\frac{1}{x(x^2-1)logx}$[4][5].

Because $J(x)$ is a step function, there is a jump $\frac{1}{m}$ at $x = p_n^m$, and at other real numbers are fixed values, so

$$J'(x) = \frac{1}{m}\delta(x - p_n^m); n = 1, 2, 3, ...; m = 1, 2, 3, ... \tag{21}$$

where p_n is the n^{th} prime.The $\delta(x)$ function is defined by the following two expressions[1],

$$\delta(x - p_n^m) = \begin{cases} \infty, & x = p_n^m; \\ 0, & x \neq p_n^m. \end{cases} \tag{22}$$

and

$$\int_a^b \delta(x - p_n^m)dx = \begin{cases} 0, & \text{a,b} > p_n^m\text{, or a,b} < p_n^m; \\ 1, & a < p_n^m < b. \end{cases} \tag{23}$$

Connecting Formula (19) and (21), we obtain some properties of nontrivial zeros.

(1)When $x = p_n^m$,$J'(p_n^m) = \infty$,we can get

$$\psi(p_n^m) = \sum_{n=1}^{\infty} cos(mt_n logp_n) = -\infty, n = 1, 2, 3, ...; m = 1, 2, 3, ... \tag{24}$$

In order to intuitively see the properties of the δ-function of this derivative,we use the known nontrivial zeros to calculate $J'(x)$ in Formula (19) and get Figure 1 and 2.

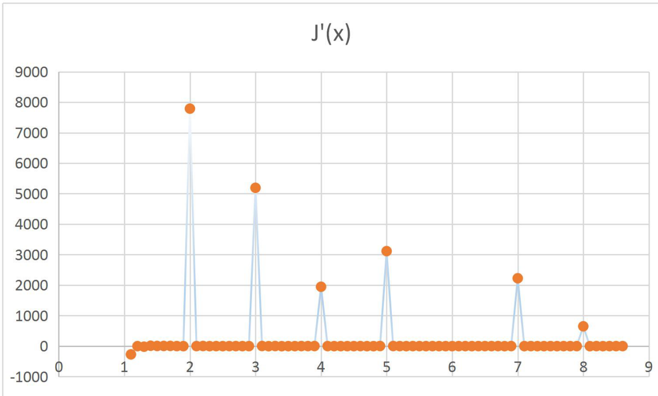

Figure 1: $J'(x)$, $x \in [1.1, 8.6]$

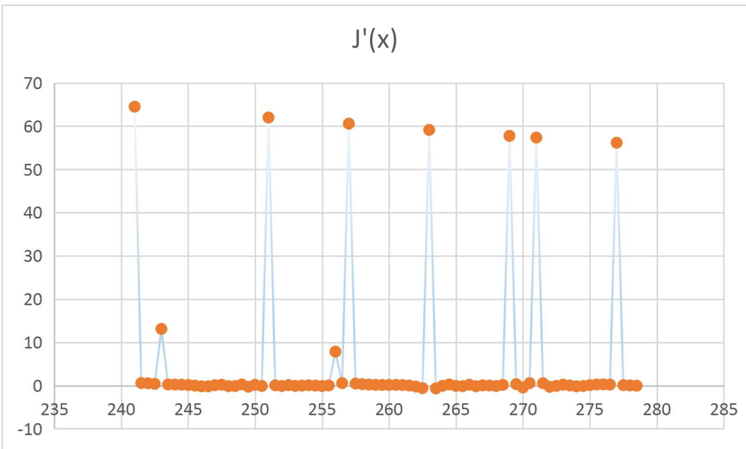

Figure 2: $J'(x)$,$x \in [241, 278]$

Figures 1 and 2 show the calculation results by substituting the known 62000 zeros[3] into Formula (19).We know that the number of zeros is on the order of

10^{13}.The reason why I chose 62000 zeros is that my computer can only calculate so many zeros. We calculate it in Excel. Figure 1 is the result of $J'(x)$ calculated every $\Delta x = 0.1$ in the interval of $x \in [1.1, 8.6]$. Figure 2 is the result of $J'(x)$ calculated every $\Delta x = 0.5$ in the interval of $x \in [241, 278]$.The points of the graph is the result of calculation, and the lines between the points indicate the trend of change.

As can be seen from Figure 1, the first high points are $x = 2, 3, 5, 7$, they are primes. The first secondary high point on the left, $x = 4 = 2^2$. The second secondary high point, $x = 8 = 2^3$. The value of $J'(x)$ of other points is almost 0 compared with the value of prime. With the increase of the number of zeros, the value of $J'(x)$ will increase. If the number of zeros tends to infinity, the value of prime and its power tend to infinity. This is what the δ-function means.If more zeros are used, the δ-function of derivative $J'(x)$ is better.

As can be seen from Figure 2, the first high points are $x = 241, 251, 257, 263, 269$ and $271, 277$,they are primes. The first secondary high point on the left, $x = 243 = 3^5$. The second secondary high point, $x = 256 = 2^8$. The value of $J'(x)$ of other points is almost 0 compared with the value of primes and their power.

This provides us with a way to judge primes by zeros of Riemann's $\zeta(s)$ function. For any odd number x, calculate $\psi(x)$. If $\frac{\psi(x)}{\psi(x+0.1)}$ is a large number, then the number is a prime or the power of a prime. Otherwise, it's a composite number.

(2)When $x \neq p_n^m$,$J'(x) = 0$,we can get

$$\frac{1}{\sqrt{x}} \sum_{n=1}^{\infty} 2cos(t_n logx) = 1 - \frac{1}{x(x^2-1)}; x > 1. \tag{25}$$

For example,when $x = e$,$J'(e) = 0$,according to the Formula (25) we can get

$$\sum_{n=1}^{\infty} cost_n = \frac{\sqrt{e}}{2}(1 - \frac{1}{e(e^2-1)}) = 0.776894261... = \lambda_1. \tag{26}$$

This is the formula which is the sum of cosine of all nontrivial zeros. In order to study the convergence of this sum, the following function is introduced,

$$\lambda(N) = \sum_{n=1}^{N} cost_n, \tag{27}$$

then the $\lambda(N)$ can be calculated according to the known nontrivial zeros.See figure 3 and 4.

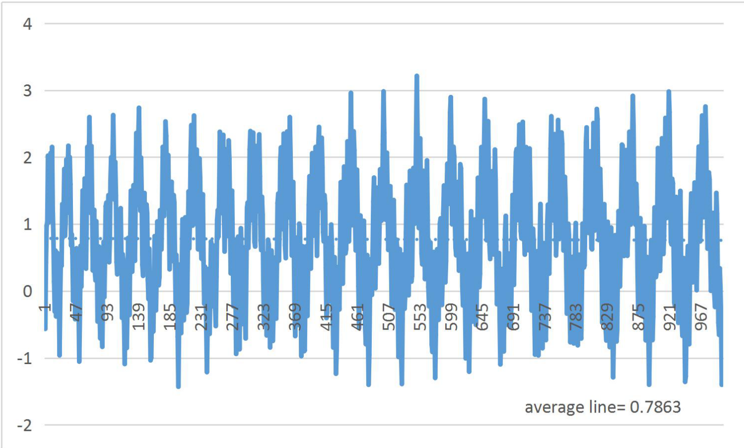

Figure 3: $\lambda(N)$, $(N < 1000)$

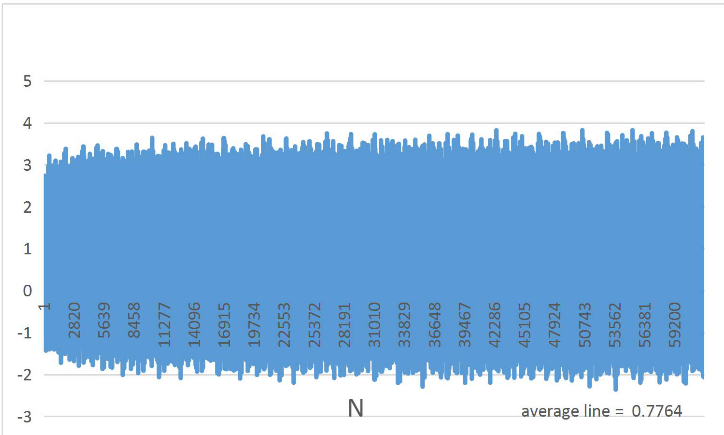

Figure 4: $\lambda(N)$,$(N < 62000)$

So,$\lambda(N)$ oscillates back and forth around λ and $\lambda_1 = \lim_{N\to\infty}\lambda(N)$.Since

$$\lim_{n\to\infty}\frac{cost_{n+1}}{cost_n} \neq 0, \tag{28}$$

the summation of Formula (26) is not absolutely convergent. Figure 3 shows the calculation results of the first 1000 nontrivial zeros, which shows that $\lambda(N)$ fluctuates back and forth around $\overline{\lambda} = 0.7863$, and has a certain regularity. As the number of nontrivial zeros increases,$\lambda(N)$ tends to $\overline{\lambda}$. Figure 4 shows the calculation results of the first 62000 nontrivial zeros. At this time, $\overline{\lambda} = 0.7764$, which is very close to $\lambda_1 = 0.7768....$ This shows that the sum of cosines of all nontrivial zeros is a global property. By extension, we obtain the cosine of m times nontrivial zeros below.

When $x = e^m$,$m = 1, 2, 3, ...$,$J'(e^m) = 0$,according to the Formula (19) we can get the sum of cosine of m times nontrivial zeros

$$\sum_{n=1}^{\infty} cos(mt_n) = \frac{e^{\frac{m}{2}}}{2}(1 - \frac{1}{e^m(e^{2m}-1)}) = \lambda_m. \tag{29}$$

We can calculate some of them below,

$$\sum_{n=1}^{\infty} cos(2t_n) = 1.35570... = \lambda_2, \tag{30}$$

$$\sum_{n=1}^{\infty} cos(3t_n) = 2.24056... = \lambda_3, \tag{31}$$

$$\sum_{n=1}^{\infty} cos(4t_n) = 3.69450... = \lambda_4, \tag{32}$$

$$\sum_{n=1}^{\infty} cos(5t_n) = 6.09124... = \lambda_5. \tag{33}$$

(3)As can be seen from Figure 1, when $x \in (1, 2)$,$J'(x) = 0$, which is in good agreement with Formula (19). But when $x < 1$, for example,When $x = p_n^{-m} < 1$,because $cos[t_n log p_n^{-m}] = cos[t_n log p_n^{m}]$,we have

$$J'(p_n^{-m}) \neq 0. \tag{34}$$

We find that there are many no zero points of $J'(x)$ in $x \in (0, 1)$.When Riemann defined the step function, $J(x) = 0$, $x \in (0, 1)$.Therefore, $J'(x) = 0$, $x \in (0, 1)$.This shows that the step function inverted by Riemann's zeta function,$\zeta(s)$, is not consistent with Riemann's original assumption in $x \in (0, 1)$. There are very rich structure in $x \in (0, 1)$.In Section 2,we will study interval (0,1) and show the step function of this interval and its derivative.

2 The step function $J^{-}(x)$ in $x \in (0,1)$

In Riemann's paper[4],he suppose $J(x) = 0; x < 2$,i.e.,

$$J(x) = 0, x \in (0,1). \tag{35}$$

According to the Formula (35),we have

$$J'(x) = 0, x \in (0,1). \tag{36}$$

But the derivative of $J(x)$ calculated by the Formula (19) is not 0, which indicates that the Formula (1) is not valid in the interval where x is less than 1,and the Riemann's suppose(35) is wrong.

2.1 The reason why Riemann's step function does not hold in the interval $x \in (0,1)$.

From Euler's formula[4],

$$\frac{log\zeta(s)}{s} = \int_1^{\infty} J(x)x^{-s-1}dx, \tag{37}$$

it hold only if $x > 1$.

Let

$$u = logx, \tag{38}$$

$$a + i\omega = s, \tag{39}$$

where a is constant,a and ω are real.So Formula (37) becomes

$$\frac{log\zeta(a+i\omega)}{a+i\omega} = \int_0^{\infty} J(u)e^{-au}e^{-iwu}du = \int_0^{\infty} \Phi(u)e^{-iwu}du, \tag{40}$$

it hold only if $u > 0$.Where $\Phi(u) = J(u)e^{-au}$.

According to the Fourier Theorem,The standard Fourier transform[1] is

$$\Phi(\omega) = \int_{-\infty}^{\infty} \Phi(u)e^{-iwu}du, \tag{41}$$

integral interval is $+\infty > u > -\infty$. In this way, the $\Phi(u)$ derived from Fourier inversion,

$$\Phi(u) = \frac{1}{2\pi}\int_{ai-\infty}^{ai+\infty} \Phi(\omega)e^{iwu}d\omega, \tag{42}$$

it hold for $+\infty > u > -\infty$.Here, the selection of a is required that $e^{-ua} \to 0$ when $u \to \infty$. Therefore,when $u \in (0,\infty)$,$a > 0$ is required.When $u \in (-\infty, 0)$,$a < 0$ is required.

Now,

$$\Phi(\omega) = \frac{log\zeta(a+i\omega)}{a+i\omega}, \tag{43}$$

is the Fourier form of zeta function found by Riemann.Substituting $\Phi(\omega)$ into Fourier transformation equation (42),the result of transformation $\Phi(u) = J(u)e^{-au}$ is not only established in the interval where $u > 0$(i.e,$x > 1$), but also in the interval $u < 0$(i.e,$x \in (0,1)$).However, Riemann only discussed the case where u is greater than 0(i.e,$x > 1$) and obtained Formula (1).We discuss the case where u is less than 0(i.e,$x \in (0,1)$) now.

2.2 The step function $J^-(x)$ in $x \in (0,1)$.

When $u \in (-\infty, 0)$,i.e,$x \in (0,1)$,let $y = \frac{1}{x}$,the Formula (42) becomes

$$J(y) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{log\zeta(s)}{s} y^{-s} ds, \tag{44}$$

where

$$\xi(s) = \Gamma(\frac{s}{2}+1)(s-1)\pi^{-\frac{s}{2}}\zeta(s), \tag{45}$$

the zeros of $\xi(s)$ is same as the nontrivial zeros of $\zeta(s)$,and we have symmetry of $\xi(s) = \xi(1-s)$,

$$\xi(s) = \xi(0)\prod_{\rho}(1-\frac{s}{\rho}), \tag{46}$$

so

$$log\zeta(s) = -log\Gamma(\frac{s}{2}+1) - log(s-1) + \frac{s}{2}log\pi + ln\xi(0) + \sum_{\rho} log(1-\frac{s}{\rho}), \tag{47}$$

and let $s' = -s$,Formula (42) becomes

$$J(y) = \frac{1}{2\pi i}\int_{-a+i\infty}^{-a-i\infty} \frac{log\zeta(s')}{s'} y^{s'} ds'; (y > 1), \tag{48}$$

substituting Formula (45) into Formula (46), integral item by item.

2.2.1 $Li^-(x)$

$$Li^-(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{-log(s-1)}{s} x^s ds, x \in (0,1). \tag{49}$$

Let $y = \frac{1}{x}, s' = -s$,

$$Li^-(y) = -\frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty} \frac{-log(-s'-1)}{s'} y^{s'} ds', y \in (1,\infty). \tag{50}$$

This is equivalent to an typical integral

$$\frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty} \frac{log(1-\frac{s}{\beta})}{s} y^s ds, y \in (1,\infty), \tag{51}$$

where $\beta=-1<0$,therefore

$$Li^{-}(y)=\int_{\infty}^{y}\frac{dt}{t^2logt};y\in(1,\infty). \tag{52}$$

Therefore,

$$Li^{-}(x)=\int_{0}^{\frac{1}{x}}\frac{dt}{logt};x\in(0,1). \tag{53}$$

2.2.2 $Li^{-}(x^{\rho})$

$$Li^{-}(x^{\rho})=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{log(1-\frac{s}{\rho})}{s}x^s ds, x\in(0,1), \tag{54}$$

where $Re\beta=-Re\rho=-\frac{1}{2}<0$,we can get

$$Li^{-}(y^{\rho})=\int_{\infty}^{y}\frac{t^{-\rho-1}}{logt}dt;y\in(1,\infty). \tag{55}$$

$$Li^{-}(x^{\rho})=\int_{0}^{\frac{1}{x}}\frac{t^{\rho}}{tlogt}dt;x\in(0,1). \tag{56}$$

2.2.3 $J_{\Gamma}^{-}(x)$

$$J_{\Gamma}^{-}(x)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{-log\Gamma(\frac{s}{2}+1)}{s}x^s ds, x\in(0,1), \tag{57}$$

we can get

$$J_{\Gamma}^{-}(x)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\sum_{n=1}^{\infty}\frac{d[\frac{log(1+\frac{s}{2n})}{s}]}{ds}x^s ds, x\in(0,1), \tag{58}$$

$$J_{\Gamma}^{-}(y)=-\int_{0}^{y}\frac{1}{t(1-t^2)logt}dt;y\in(1,\infty), \tag{59}$$

$$J_{\Gamma}^{-}(x)=\int_{\frac{1}{x}}^{\infty}\frac{t}{(t^2-1)logt}dt;x\in(0,1). \tag{60}$$

Therefore,the step function $J^{-}(x)$ in the interval $x\in(0,1)$ obtained from zeta function through Fourier transformation is

$$J^{-}(x)=Li^{-}(x)-\sum_{\rho}[Li^{-}(x^{\rho})+Li^{-}(x^{1-\rho})]-log2+\int_{\frac{1}{x}}^{\infty}\frac{t}{(t^2-1)logt}dt;x\in(0,1). \tag{61}$$

Obviously $J^{-}(x)$ is different to the Formula (1).

2.3 The derivative and image of $J^{-}(x)$.

Let's calculate the derivative of $J^{-}(x)$ first.We calculate directly,

$$Li^{-'}(x) = \frac{1}{x^2 logx}, \tag{62}$$

$$Li^{-'}(x^{\rho}) = \frac{x^{-\rho}}{xlogx}, \tag{63}$$

$$Li^{-'}(x^{1-\rho}) = \frac{x^{\rho-1}}{xlogx}. \tag{64}$$

Therefore,because $\rho = \frac{1}{2} \pm t_n i$,the derivative of $J^{-}(x)$ is

$$J^{-'}(x) = \frac{1}{x^2 logx}[1 - \frac{x^3}{1-x^2} - \sqrt{x}\sum_{n=1}^{\infty} 2cos(t_n logx)]; x \in (0,1). \tag{65}$$

It is also different to the Formula (19).Above Formula (65) can be rewritten by

$$J^{-'}(x) = -\frac{1}{x^2}\{\frac{1}{log(\frac{1}{x})}[1 - \frac{1}{\frac{1}{x}((\frac{1}{x})^2 - 1)} - \frac{1}{\sqrt{\frac{1}{x}}}\sum_{n=1}^{\infty} 2cos[t_n log(\frac{1}{x})]\}; x \in (0,1). \tag{66}$$

According to Formula (21),

$$J^{-'}(x) = -\frac{1}{x^2}\{\frac{1}{m}\delta(\frac{1}{x} - p_n^m)\}. \tag{67}$$

It is easy to prove

$$\int_{p_n^{-m}-\epsilon}^{p_n^{-m}+\epsilon} J^{-'}(x)dx = -\frac{1}{m}, 0 < \epsilon < 1, \tag{68}$$

it shows that $J^{-}(x)$ is a step decreasing function, jumping $-\frac{1}{m}$ at $x = \frac{1}{p_n^m}$.According to the characteristics of $J^{-'}(x)$,we can draw the image of $J^{-}(x)$, Figure 5.

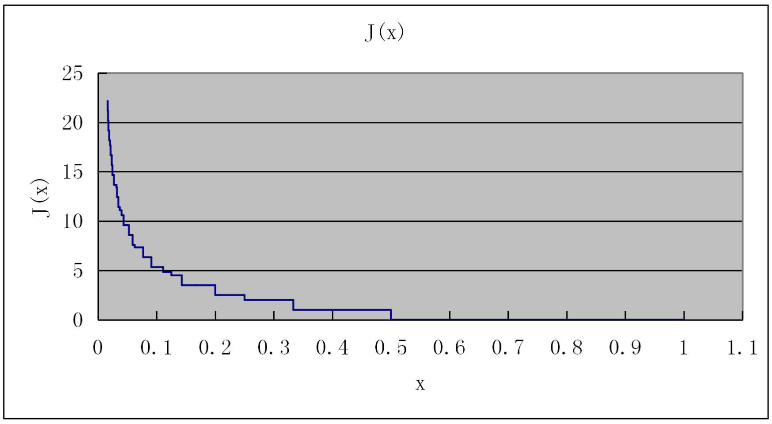

Figure 5: The step function $J^-(x)$ for $x \in (0,1)$

When $x = \frac{1}{e}$,$J^{-'}(\frac{1}{e}) = 0$,we can get same equation from Formula (65),

$$\sum_{n=1}^{\infty} cost_n = \frac{\sqrt{e}}{2}(1 - \frac{1}{e(e^2-1)}) = 0.776894261... = \lambda_1. \tag{69}$$

This shows the rationality of Formula (65).

Let $y = \frac{1}{x}$,the Formula (65) becomes

$$J^{-'}(y) = \frac{dJ^-(y)}{dy} = \frac{1}{logy}[1 - \frac{1}{y(y^2-1)} - \frac{1}{\sqrt{y}}\sum_{n=1}^{\infty} 2cos(t_n logy)]; y \in (1,\infty). \tag{70}$$

It is same with formula (19).Therefore,$J^-(x)$ and $J(x)$ have the symmetry of x to $\frac{1}{x}$ transformation.

3 Density function of prime pair of even

With the derivative $J'(x)$ of Riemann's step function, we can calculate the number of even decomposed into prime pairs. Through the relationship formula (2) between prime distribution function and Riemann's step function,we can get the derivative of $\pi(x)$,

$$\pi'(x) = \sum_n \frac{\mu(n)}{n^2} x^{\frac{1}{n}-1} J'(x^{\frac{1}{n}}). \tag{71}$$

The first two items are

$$\pi'(x) = J'(x) - \frac{1}{2^2}\frac{1}{\sqrt{x}}J'(\sqrt{x}) + ... \tag{72}$$

The $\pi(x)$ is a step function with a step of 1. There are jumps only at $x = p_n$, 1 for each jump,

$$\pi'(x) = \delta(x - p_n). \tag{73}$$

The derivative is infinite only when x is a prime, and it is 0 everywhere else. Especially at $x = p_n^m, m > 1$, the infinity of $J'(x)$ is cancelled by multiple summations.

Due to the complexity of Formula (71),it is not possible to obtain a concise expression of $\pi'(x)$ expressed by nontrivial zeros. But this does not hinder our next research.

For any number $E \in R^+$,let

$$\rho(E, x) = \pi'(x)\pi'(E - x); x \in (1, E - 1), \tag{74}$$

where $\pi(x)$ is the prime distribution in the positive direction of x-axis and $\pi(E-x)$ is the prime distribution in the negative direction of x-axis after translating E(See Figure 7).As you can see from the integral below, $\rho(E, x)$ here is the density.

According to Formula (73),we have

$$\rho(E, x) = \delta(x - p_n)\delta(E - x - p_m); x \in (1, E - 1), \tag{75}$$

where $p_n, p_m < E$.Obviously, only when $E = p_n + p_m$,$\rho(E, x)$ may not be zero.

So,

$$\int_1^{E-1} \rho(E, x)dx = \sum_{p_n, p_m < E} \delta(E - (p_n + p_m)) \equiv N(E). \tag{76}$$

That is, $N(E)$ is the number of pair (p_n, p_m) that the E can be expressed as the sum of p_n and p_m,i.e.,$E = p_n + p_m$.

(1)When E is a real but not even or odd,$N(E) = 0$.

(2)When E is an odd,$N(E) \neq 0$ only one of p_n and p_m is $p_1 = 2$.

(3)When E is an even,Goldbach conjectures that $N(E)$ is greater than 1 for $E \geq 6$. First, we can count the $N(E)$ before an even number E. Figure 6 shows the $N(E)$ value of E less than 308.

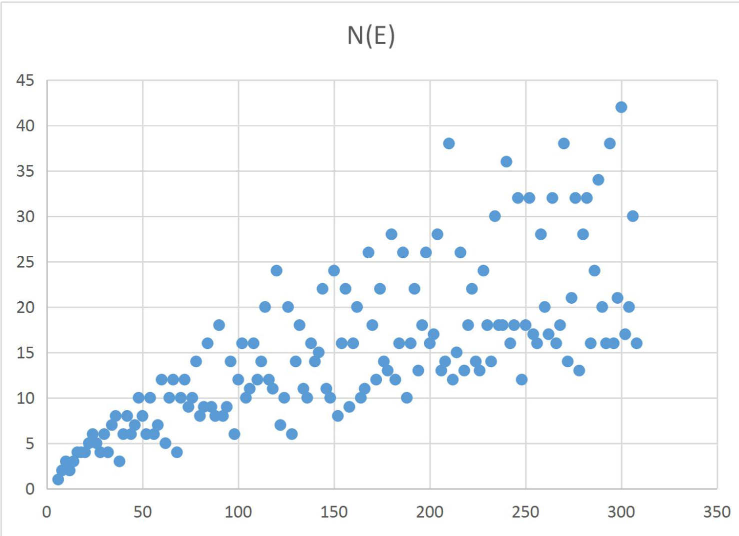

Figure 6: $N(E)$ of Even number E;$E \leq 308$

Second,there are two ways to understand $N(E)$. Mark the prime number p_n in $x \in (0, E)$ at the corresponding point on the x-axis.(1) On the negative direction of x-axis,mark $-p_n$ at the corresponding point.After translating E the x-axis become x'-axis.We have

$$x' = x + E. \tag{77}$$

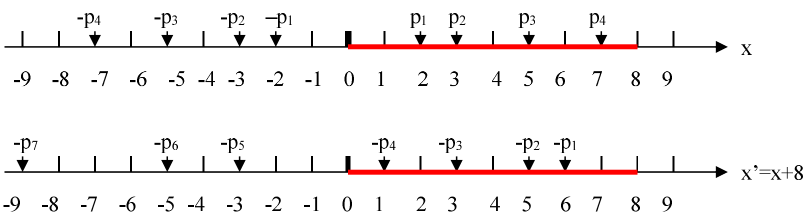

Figure 7: x-axis and x'-axis

Then the number met together of primes on the x-axis and the x'-axis is $N(E)$.In Figure 7, p_2 meets $-p_3$, p_3 meets $-p_2$, so $N(E) = 2$.(2) The prime sequence in $(0, E)$ is folded at $\frac{E}{2}$.If $\frac{E}{2}$ is not a prime, then two times of the number of prime met together is $N(E)$.In Figure 7,$E/2 = 4$,4 is not prime. After folding, p_2 meets p_3, so $N(E) = 2 \times 1 = 2$. If $\frac{E}{2}$ is a prime, then it is 2 times of the number of prime met together +1.

Third,according the derivative of Riemann's step function,we get the express of $N(E)$,

$$N(E) = \int_{1}^{E-1} \pi'(x)\pi'(E-x)dx; x \in (1, E-1). \tag{78}$$

According to the Formula (71) of $\pi'(x)$,

$$N(E) = \int_{1}^{E-1} \sum_{n_1=1}^{M(x)} \frac{\mu(n_1)}{n_1^2} x^{\frac{1}{n_1}-1} J'(x^{\frac{1}{n_1}}) \sum_{n_2=1}^{M(E-x)} \frac{\mu(n_2)}{n_2^2} (E-x)^{\frac{1}{n_2}-1} J'((E-x)^{\frac{1}{n_2}}) dx. \tag{79}$$

This is a complicated calculation formula.Compared with the complexity of Figure 6, its expression Formula (79) is a big development.The above Formula (79) provides us with a new way to prove Goldbach's conjecture.We only need to prove $N(E) > 1$ for every even E to prove Goldbach's conjecture. Therefore, we can simplify Formula (79) to prove $N(E) > 1$ in four steps as follows.

3.1 The integral interval is reduced from $(1, E-1)$ to $(2^-, E-2^-)$.

Here $2^- = 2 - \epsilon$,$\epsilon \to 0$.Because $J'(x) = 0, x \in (1, 2)$,

$$N(E) = \int_{2^-}^{E-2^-} \sum_{n_1=1}^{M(x)} \frac{\mu(n_1)}{n_1^2} x^{\frac{1}{n_1}-1} J'(x^{\frac{1}{n_1}}) \sum_{n_2=1}^{M(E-x)} \frac{\mu(n_2)}{n_2^2} (E-x)^{\frac{1}{n_2}-1} J'((E-x)^{\frac{1}{n_2}}) dx. \tag{80}$$

The advantage of this method is to avoid the singularity of $J'(x)$ near $x = 1$.

3.2 $N(E)$ can be simplified to the relationship with $N_J(E)$.

Because $J(x)$ and $\pi(x)$ are simple increasing function,$J'(x)$ and $\pi'(x)$ are always positive numbers.Therefore,let all $\mu(n) = -1$,we have

$$\pi'(x) > J'(x) - \sum_{n_1=2}^{M(x)} \frac{1}{n_1^2} x^{\frac{1}{n_1}-1} J'(x^{\frac{1}{n_1}}), \tag{81}$$

$$\pi'(E-x) > J'(E-x) - \sum_{n_2=2}^{M(E-x)} \frac{1}{n_2^2} (E-x)^{\frac{1}{n_2}-1} J'((E-x)^{\frac{1}{n_2}}). \tag{82}$$

Because

$$\int_{2^-}^{E-2^-} J'(x)J'(x^{\frac{1}{n}})dx < \int_{2^-}^{E-2^-} J'(x)J'(x)dx, \tag{83}$$

Therefore,

$$N(E) > \int_{2^-}^{E-2^-} [1-\sum_{n_1=2}^{M(x)} \frac{1}{n_1^2} x^{\frac{1}{n_1}-1}]J'(x)[1-\sum_{n_2=2}^{M(E-x)} \frac{1}{n_2^2}(E-x)^{\frac{1}{n_2}-1}]J'(E-x)dx. \quad (84)$$

Because

$$x^{\frac{1}{n}-1} < 1; (E-x)^{\frac{1}{n}-1} < 1; for n \geq 2, x > 1, \quad (85)$$

So

$$N(E) > \int_{2^-}^{E-2^-} [1-\sum_{n_1=2}^{M(x)} \frac{1}{n_1^2}]J'(x)[1-\sum_{n_2=2}^{M(E-x)} \frac{1}{n_2^2}]J'(E-x)dx. \quad (86)$$

Because

$$\sum_{n_1=2}^{M(x)} \frac{1}{n_1^2} < \zeta(2) - 1 = \frac{\pi^2}{6} - 1 = \chi, \quad (87)$$

we have

$$N(E) > (1-\chi)^2 \int_{2^-}^{E-2^-} J'(x)J'(E-x)dx. \quad (88)$$

Let

$$N_J(E) = \int_{2^-}^{E-2^-} J'(x)J'(E-x)dx, \quad (89)$$

which can be calculated directly.We have the relation

$$N(E) > (1-\chi)^2 N_J(E). \quad (90)$$

3.3 Estimation of $N_J(E)$.

Substituting $J'(x)$ of Formula (19) in the Formula (89),

$$\begin{aligned} N_J(E) &= \int_{2^-}^{E-2^-} \frac{1}{logxlog(E-x)}[1-\frac{1}{x(x^2-1)} - \frac{1}{\sqrt{x}}\sum_{n=1}^{\infty} 2cos(t_n logx)] \\ &\quad [1-\frac{1}{(E-x)((E-x)^2-1)} - \frac{1}{\sqrt{(E-x)}}\sum_{n=1}^{\infty} 2cos(t_n log(E-x))]dx \\ &. \end{aligned} \quad (91)$$

This integral can be divided into four parts,

$$N_J(E) = N_{J1}(E) + N_{J2}(E) + N_{J3}(E) + N_{J4}(E), \quad (92)$$

where

$$N_{J1}(E) = \int_{2^-}^{E-2^-} \frac{1}{logxlog(E-x)}[1-\frac{1}{x(x^2-1)}][1-\frac{1}{(E-x)((E-x)^2-1)}]dx, \quad (93)$$

$$N_{J2}(E) = -\int_{2^-}^{E-2^-} \frac{1}{logxlog(E-x)}[1-\frac{1}{x(x^2-1)}]\frac{1}{\sqrt{E-x}}\sum_{n=1}^{\infty} 2cos(t_n log(E-x))dx, \tag{94}$$

$$N_{J3}(E) = -\int_{2^-}^{E-2^-} \frac{1}{logxlog(E-x)}[1-\frac{1}{(E-x)((E-x)^2-1)}]\frac{1}{\sqrt{x}}\sum_{n=1}^{\infty} 2cos(t_n logx)dx, \tag{95}$$

$$N_{J4}(E) = \int_{2^-}^{E-2^-} \frac{1}{logxlog(E-x)\sqrt{x(E-x)}}\sum_{n=1}^{\infty} 2cos(t_n logx)\sum_{n=1}^{\infty} 2cos(t_n log(E-x))dx, \tag{96}$$

Because in $x \in (2^-, E-2^-)$

$$\frac{1}{logxlog(E-x)} > \frac{1}{(log\frac{E}{2})^2}, \tag{97}$$

$$1 - \frac{1}{x(x^2-1)} > \frac{5}{6}, \tag{98}$$

$$1 - \frac{1}{(E-x)((E-x)^2-1)} > \frac{5}{6}. \tag{99}$$

We can calculate directly,

$$N_{J1}(E) > (\frac{5}{6})^2\frac{E-4}{(log\frac{E}{2})^2}. \tag{100}$$

Do $x \to E-x$ transform,we get

$$N_{J3}(E) = N_{J2}(E). \tag{101}$$

Because of

$$\int_{2^-}^{E-2^-} J'(x)dx = J(E), \tag{102}$$

according to the Formula (98),

$$\int_{2^-}^{E-2^-} \frac{1}{logx}[1-\frac{1}{x(x^2-1)}]dx > \int_{2^-}^{E-2^-} \frac{5}{6}\frac{1}{logx}dx > \frac{5}{6}\frac{E-4}{log\frac{E}{2}}, \tag{103}$$

we have

$$-\int_{2^-}^{E-2^-} \frac{1}{\sqrt{x}logx}\sum_{n=1}^{\infty} 2cos(t_n logx)dx > J(E) - \frac{5}{6}\frac{E-4}{log\frac{E}{2}}. \tag{104}$$

Therefore,

$$N_{J3}(E)+N_{J2}(E)>2[\frac{J(E)}{logE}-\frac{5}{6}\frac{E-4}{log\frac{E}{2}logE}], \tag{105}$$

Because of

$$J(E)>\frac{E}{logE}, \tag{106}$$

therefore,

$$N_{J3}(E)+N_{J2}(E)>2[\frac{E}{(logE)^2}-\frac{5}{6}\frac{E-4}{log\frac{E}{2}logE}]. \tag{107}$$

Because of $J'(x)>0$,we have

$$-\frac{1}{\sqrt{x}}\sum_{n=1}^{\infty}2cos(t_n logx)>-(1-\frac{1}{x(x^2-1)}), \tag{108}$$

$$-\frac{1}{\sqrt{E-x}}\sum_{n=1}^{\infty}2cos(t_n log(E-x))>-(1-\frac{1}{(E-x)((E-x)^2-1)}), \tag{109}$$

thus

$$N_{J4}(E)>(\frac{5}{6})^2\frac{E-4}{(log\frac{E}{2})^2}. \tag{110}$$

Therefore

$$N_J(E)>2(\frac{5}{6})^2\frac{E-4}{(log\frac{E}{2})^2}+2[\frac{E}{(logE)^2}-\frac{5}{6}\frac{E-4}{log\frac{E}{2}logE}]. \tag{111}$$

3.4 Lower limit of $N(E)$.

According to the Formula (90) and (111),we get the lower limit of $N(E)$ for every $E\geq 6$,

$$N(E)>0.355\{2(\frac{5}{6})^2\frac{E-4}{(log\frac{E}{2})^2}+2[\frac{E}{(logE)^2}-\frac{5}{6}\frac{E-4}{log\frac{E}{2}logE}]\};E\geq 6. \tag{112}$$

If $E\gg 6$,$E-4\approx E$,$log\frac{E}{2}\simeq logE$,

$$N(E)>0.611\frac{E}{(logE)^2}. \tag{113}$$

The lower limit of this $N(E)$ is drawn in Figure 6, that is Figure 8 below.

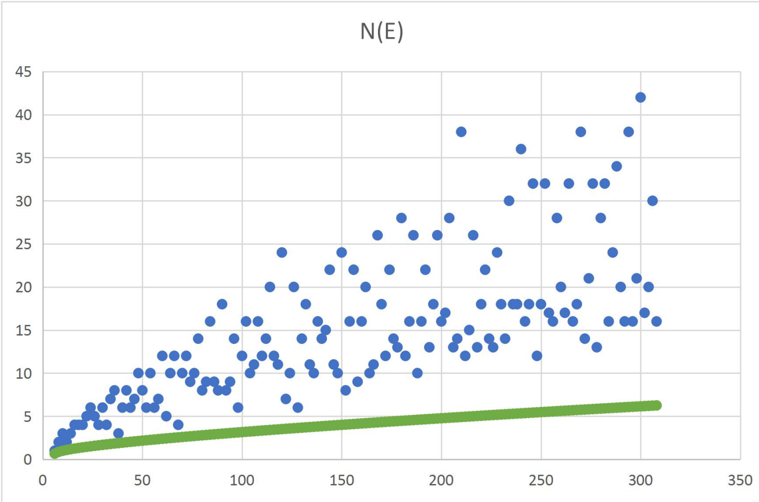

Figure 8: $N(E)$ and its lower limit;$E \leq 308$

It is directly calculated in the Formula (112) that for all even numbers $E \geq 6$,we get

$$N(E) > 1.5 > 1; E \geq 6. \tag{114}$$

This shows that the number of any even number greater than 6 expressed as the sum of prime pairs is greater than 1.This is Goldbach's conjecture.

Finally, we want to emphasize that the prime step function derived from the Riemann's zeta function in Fourier Theorem are

$$J(x) = Li(x) - \sum_{n=1}^{\infty}[Li(x^{\frac{1}{2}\pm it_n}) - log2 + \int_x^{\infty} \frac{dt}{t(t^2-1)logt}; x > 1, \tag{115}$$

$$J^-(x) = Li^-(x) - \sum_{n=1}^{\infty}[Li^-(x^{\frac{1}{2}\pm it_n}) - log2 + \int_{\frac{1}{x}}^{\infty} \frac{tdt}{(t^2-1)logt}; x \in (0,1). \tag{116}$$

They are exact expressions.

The distribution function of prime is

$$\pi(x) = \sum_{n=1}^{M} \frac{\mu(n)}{n} J(x^{\frac{1}{n}}); x > 1, \tag{117}$$

and

$$\pi^-(x) = \sum_{n=1}^{M} \frac{\mu(n)}{n} J^-(x^{\frac{1}{n}}); x \in (0,1), \tag{118}$$

which is the distribution function of reciprocal prime.Where $(\frac{1}{x})^{\frac{1}{M}} \geq 2$.

So, the left first prime p_n of any real number $x > 1$,

$$n = \pi(x). \tag{119}$$

$$p_n = \min\{\pi^{-1}(n)\}. \tag{120}$$

Therefore, we have the analytic expression of all primes.

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